BAULKHAM HILLS HIGH SCHOOL

YEAR 11

HSC ASSESSMENT TASK

December 2008

MATHEMATICS
EXTENSION 1

Time allowed - 70 minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Allocated marks are indicated for each question.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet provided.

Question 1

a) Solve $\cos 2x = \sin x$ for $0 \le x \le 2\pi$ **4**

b) (i) Find $\int \frac{1}{4(2x+1)^3}\,dx$ **2**

(ii) Find $\int \sin^2 4x\,dx$ **2**

Question 2

Differentiate $x\sin x$. Hence or otherwise find $\int x\cos x\,dx$ **3**

Question 3

Find the Cartesian equation of the curve with parametric equations:

a) $x = t^2 + 2,\ y = 3t$ **1**

b) $x = \cos t\ ,\ y = \sin 2t$ **2**

Question 4

If α, β and γ are the roots of $2x^3 - 5x^2 + 3x - 15 = 0$, find the value of

a) $\alpha + \beta + \gamma$ **1**

b) $\alpha\beta\gamma$ **1**

c) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **2**

Question 5

Find $\lim\limits_{x \to 0} \frac{1 - \cos 2x}{x^2}$, showing all necessary working **2**

Question 6

If $P(x) = x^3 - 3x + 2$

a) Factorise $P(x)$ completely **3**

b) Solve $x^3 - 3x + 2 > 0$ **2**

Question 7

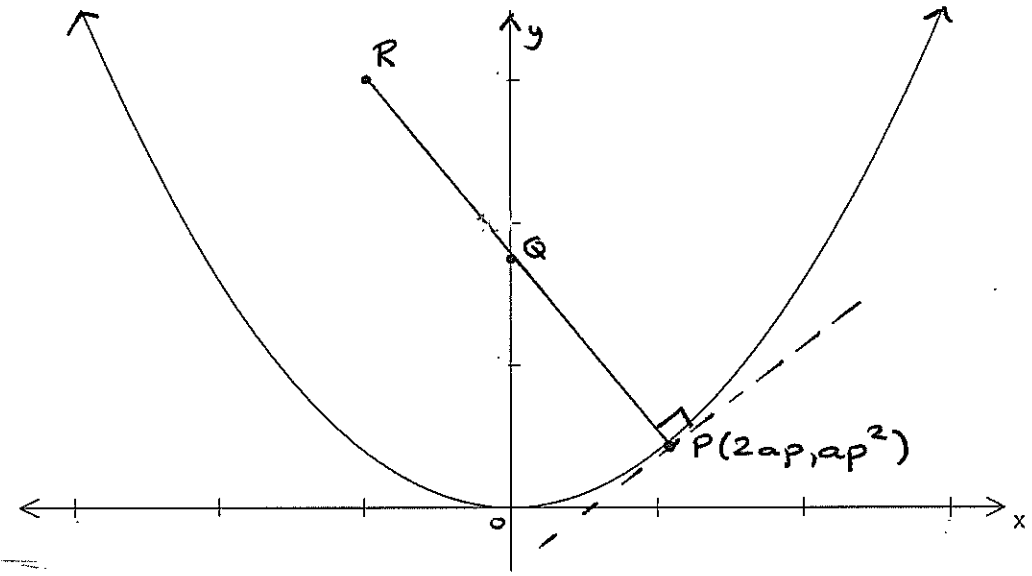
Given that $P(x) = x^3 + ax^2 + bx + c$ and

* $P(0) = 1$
* $x - 1$ is a factor of $P(x) + 1$
* $x + 1$ is a factor of $P(x) - 1$

Determine the values for *a, b* and *c* **4**

Question 8

The point $P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P cuts the y-axis at Q. PQ is then produced to R such that PQ=QR.



a) Show that the equation of the normal at P is $x + py = ap^3 + 2ap$ **2**

b) Find the coordinates of Q and R **3**

c) Deduce that the locus of R is a parabola and state the coordinates of its vertex. **3**

Question 9

The area between the curve $y = 1 + 2\cos x$, the x-axis and the lines $x = 0, x = \frac{\pi}{2}$ is rotated about the x-axis. Find the exact volume of the solid of revolution generated. **4**

Question 10

The cubic $y = x^3 + ax^2 + bx + c$ has three distinct roots α, β and γ which form an arithmetic progression, with $\alpha < \beta < \gamma$.

Given that $\beta = k$, find the coordinates of the point of inflexion of the curve, in simplest form. **3**

End of exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; \ x \neq 0, \ if \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \ x > 0$

Q1.

a) $1 - 2\sin^2 x = \sin x$ 1

$0 = 2\sin^2 x + \sin x - 1$

$0 = (2\sin x - 1)(\sin x + 1)$ 1

$\sin x = \frac{1}{2}$ $\sin x = -1$

1

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ 1

b) i) $\frac{1}{4}\int (2x+1)^{-3} . dx$

$= \frac{1}{4} \cdot \frac{(2x+1)^{-2}}{-2 \times 2} + c$

$= -\frac{1}{16} \cdot (2x+1)^{-2} + c$

or $\frac{-1}{16(2x+1)^2} + c$ 2 (-1 each error).(ignore c)

ii) $\int \sin^2 4x . dx$ $\left[\sin^2 A = \frac{1}{2}(1 - \cos 2A)\right]$

$= \int \frac{1}{2}(1 - \cos 8x) . dx$ 1

$= \frac{1}{2}\left(x - \frac{1}{8}\sin 8x\right) + c$ 1 (ignore c)

Q2.

$\frac{d}{dx}(x \sin x)$

$= x . \cos x + \sin x . 1$

$= x \cos x + \sin x$ 1

$\int x \cos x + \sin x . dx = x \sin x + c$ 1

$\int x \cos x . dx + \int \sin x . dx = x \sin x + c$

$\int x \cos x . dx = x \sin x - \int \sin x . dx$

$= x \sin x + \cos x + c$. 1

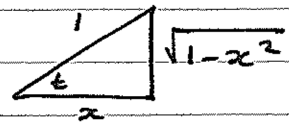
Q3.

a) $t = \frac{y}{3}$

$x = \left(\frac{y}{3}\right)^2 + 2$

$x = \frac{y^2}{9} + 2.$ 1

b) $x = \cos t$



$y = 2 \sin t \cos t$ —— 1

$\therefore y = 2.\sqrt{1-x^2}\,.\,x$ or $y = 2x\sqrt{1-x^2}$ —— 1

Alternative method :

$\cos t = x$ 1

$y = 2 \sin t \cos t \quad \therefore \quad \sin t = \frac{y}{2x}$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{2x}\right)^2 + x^2 = 1$

or $\frac{y^2}{4x^2} + x^2 = 1$

or equiv. 1

Q4.

a) $\alpha + \beta + \gamma = 5/2$ 1

b) $\alpha\beta\gamma = 15/2$ 1

c) $(\alpha-1)(\beta-1)(\gamma-1) = (\alpha-1)(\beta\gamma - \beta - \gamma + 1)$

$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$

$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \alpha\gamma) + (\alpha + \beta + \gamma) - 1$

$= \frac{15}{2} - \left(\cancel{4}\frac{3}{2}\right) + \frac{5}{2} - 1$ 1

$= \frac{15}{2}.$ 1

Q5.

$$\lim_{x \to 0} \frac{1 - (1 - 2\sin^2 x)}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \leftarrow 1$$

$$= \lim_{x \to 0} 2\left(\frac{\sin x}{x}\right)^2$$

$$= 2 \times 1^2 \qquad = \underline{2} \qquad \leftarrow 1$$

(Answer only : 1 mark)

Q6.

a) $P(1) = 1^3 - 3 + 2 = 0$

$\therefore x - 1$ is a factor

One factor = 1

$$\begin{array}{r l}
 & x^2 + x - 2 \\
x - 1 \,\big) & x^3 \qquad\; - 3x + 2 \\
 & \underline{x^3 - x^2} \\
 & \quad x^2 - 3x \\
 & \quad \underline{x^2 - x} \\
 & \qquad -2x + 2 \\
 & \qquad \underline{-2x + 2} \\
 & \qquad\qquad 0
\end{array}$$

Long div / Quad factor = 1

Final line = 1

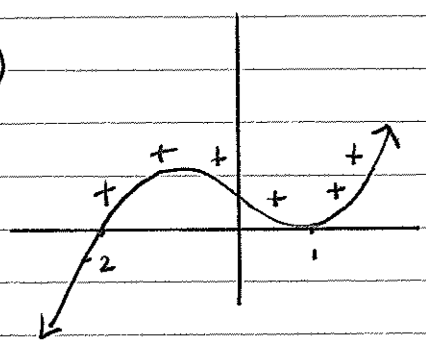
$$P(x) = (x - 1)(x^2 + x - 2)$$

$$= (x - 1)(x + 2)(x - 1)$$

$$= (x - 1)^2 (x + 2)$$

accept either. 1

b)



$x > -2$ but $x \neq 1$.

OR

$-2 < x < 1$, $x > 1$

2

Q7.

a) $P(x) = x^3 + ax^2 + bx + c$

· $P(0) = 1$: $0 + 0 + 0 + c = 1$ $\quad \underline{c = 1}$ 1

$P(x) = x^3 + ax^2 + bx + 1$

· $P(x) + 1 = x^3 + ax^2 + bx + 2$

$P(1) + 1 = 1 + a + b + 2 = 0$

$a + b = -3 \quad \ldots (1)$

· $P(x) - 1 = x^3 + ax^2 + bx$ One or both simult eqns $= 1$

$P(-1) - 1 = -1 + a - b = 0$

$a - b = 1 \quad \ldots (2)$

Solving simult : $\underline{a = -1}$, $\underline{b = -2}$

1 1

Q8.

a) $y = \dfrac{x^2}{4a}$, $\dfrac{dy}{dx} = \dfrac{2x}{4a} = \dfrac{2(2ap)}{4a}$ at P. 1

$= p$

Normal : $m = -1/p$

$y - ap^2 = -\dfrac{1}{p}(x - 2ap)$ 1

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap.$

b) At Q : $x=0$

$0 + py = ap^3 + 2ap$

$y = ap^2 + 2a \quad \therefore \; Q\,(0,\ ap^2 + 2a)$ 1

no mark for x.

At R : $x = -2ap$

$y = ap^2 + 2a + 2a = ap^2 + 4a$

$\therefore \; R\,(-2ap,\ ap^2 + 4a)$ 1 1

c) At R : $x = -2ap \quad , \; p = \dfrac{-x}{2a}$

$y = ap^2 + 4a$

$y = a\left(\dfrac{-x}{2a}\right)^2 + 4a$ 1

$$\left.\begin{aligned} y &= \frac{x^2}{4a} + 4a \\ x^2 &= 4a(y - 4a) \end{aligned}\right\} \text{a parabola.}$$ 1

Vertex $(0, 4a)$. 1

Q9. $V = \pi \displaystyle\int_0^{\pi/2} (1 + 2\cos x)^2 \cdot dx$ 1

Now $(1 + 2\cos x)^2 = 1 + 4\cos x + 4\cos^2 x$

$= 1 + 4\cos x + 4 \cdot \frac{1}{2}(1 + \cos 2x)$

$= 1 + 4\cos x + 2 + 2\cos 2x$

$= 3 + 4\cos x + 2\cos 2x$ 1

$\therefore \; V = \pi \displaystyle\int_0^{\pi/2} 3 + 4\cos x + 2\cos 2x \, . dx$

$= \pi \left[3x + 4\sin x + \sin 2x \right]_0^{\pi/2}$ 1

$$= \pi \left(\frac{3\pi}{2} + 4\underbrace{\sin\frac{\pi}{2}}_{1} + \underbrace{\sin \pi}_{0} - (0+0+0) \right)$$

$$= \pi \left(\frac{3\pi}{2} + 4 \right) \text{ units}^3$$

or $\frac{3\pi^2}{2} + 4\pi$ units^3

1

Q10.

Let roots be $\alpha = k-d$

$\beta = k$

$\gamma = k+d$

$\alpha + \beta + \gamma = 3k = -a$ 1

$k = -\frac{a}{3}$

At POI, $y'' = 0$

$y' = 3x^2 + 2ax + b$

$y'' = 6x + 2a$

$6x + 2a = 0$

$6x = -2a$

$x = -\frac{a}{3}$ 1

So the POI occurs when $x = k$, and this is one of the roots so y-value $= 0$.

$\therefore$ Coord's of POI $\underbrace{(k, 0) \text{ or } (\beta, 0)}_{1}$.